Journal of Statistical Physics, Vol. 123, No. 1, April 2006 (© 2006)
DOI: 10.1007/s10955-005-9012-8

Around Multicolour Disordered Lattice Gas

Azzouz Dermoune[1] and Servet Martinez[2]

Received June 23, 2005; accepted November 22, 2005
Published Online: April 5, 2006

We consider a system of multicolour disordered lattice gas, following closely the (monocolour) introduced by Faggionato and Martinelli[3,4]. We study the projection on the monocolour system and we derive an estimate of the closeness between grand canonical and canonical Gibbs measures.

KEY WORDS: multicolour, Gibbs measure, lattice gas.

AMS Classification: Primary: 60K35, 82C20, 82C22

1. INTRODUCTION

Faggionato and Martinelli[3,4] have studied the hydrodynamic limit for a dynamics on lattice gas which depends on a collection of real bounded i.i.d. random variables $\alpha = (\alpha_x : x \in \mathbf{Z}^d)$ corresponding to some external quenched disorder field. The hydrodynamic limit was obtained for a monocolour particles, an important byproduct of their study being the equivalence of ensembles shown in Lemma A.4.[4]

The aim of our work is to give an equivalence of ensembles result with sharp bounds for multicolour particles. Our proof uses the projection on the monocolour system, and then the equivalence of ensembles result of the monocolour lattice gas.[3] After, we study the variation of the canonical Gibbs measures of the multicolour problem with respect to the monocolour one, showing that it does not add new terms. This is done by elementary computations. We point out that the

[1] Université de Lille I, UMR-CNRS 8524, Laboratoire Paul Painlevé, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq, Cedex, France; e-mail: azzouz.dermoune@math.univ-lille1.fr

[2] Universidad de Chile, Departamento Ingeniería Matemática-Centro Modelamiento Matemático, UMR2071 CNRS-UCHILE, Casilla 170 - 3, Correo 3, Santiago, Chile; e-mail: smartine@dim.uchile.cl

equivalence of ensembles of multicolour is an important step towards obtaining the hydrodynamic limit for the multicolour case. On the other hand, as it is shown in ref. 5, the hydrodynamic limit for the multicolour case is necessary to obtain propagation of chaos.

The model introduced in Refs. 3 and 4 has been previously extended to the two colours in Refs. 1 and 2 and an equivalence of ensembles result was proven by using similar techniques as those in Ref. 3 which are different from ours.

Let us introduce the problem. $\Lambda \subset \mathbb{Z}^d$ denotes a finite set of sites, each of the sites could be empty or has a colour (or spin). The finite set of colours is denoted by I and it is useful to consider the set $I_0 = I \cup \{0\}$, where we have added the value 0 that expresses that the site has no colour (or that it is empty). We denote $|i| = \mathbf{1}(i \in I)$ for $i \in I_0$ (where we put $\mathbf{1}(x \in A)$=1 if $x \in A$, vanishing otherwise). The set of configurations is I_0^{Λ}, and its elements are denoted by $\eta = (\eta_x : x \in \Lambda)$. For each $x \in \Lambda$

$$\eta_x = \begin{cases} i & \text{if there is a particle with colour } i \in I \text{ at } x \\ 0 & \text{if there is no particle at } x \end{cases}$$

We will use the notation $|\eta| := (|\eta_x| : x \in \Lambda) \in \{0, 1\}^{\Lambda}$.

1.1. Gibbs Measures for Monocolour Lattice Gas

Let $\lambda \in \mathbb{R}$ and the disorder α be fixed. The Gibbs measure $|\mu| := \mu^{\alpha,\lambda}$ or the monocolour problem is a probability measure on $\{0, 1\}^{\Lambda}$ characterized by: all the projections $\xi \to \xi_x$, $x \in \Lambda$ are independent

$$|\mu|(\bar{\xi}) := |\mu|(\xi : \xi = \bar{\xi}) = \prod_{x\in\Lambda} |\mu|(\xi : \xi_x = \bar{\xi}_x) \quad \text{for} \quad \bar{\xi} \in \{0, 1\}^{\Lambda} ; \qquad (1.1)$$

and the marginals verify

$$|\mu|(\xi : \xi_x = 1) = \frac{e^{\lambda+\alpha_x}}{1 + e^{\lambda+\alpha_x}} . \qquad (1.2)$$

The measure $|\mu|$ is symmetric and invariant for the dynamics studied in break Refs. 3 and 4. Briefly this dynamics is given by the following rate of change $c^{\alpha}_{x,x+e}(\eta) = f_e(\alpha_x, \eta_x, \alpha_{x+e}, \eta_{x+e})$ for $\eta \in \{0, 1\}^{\Lambda}$ and $\{x, x + e\} \subset \Lambda$. Here e is an element of a symmetric neighbourhood ϵ of the origin and the functions $(f_e : e \in \epsilon)$ verify the conditions specified in Ref. 4 (symmetry, exclusion, lower uniform boundedness and detailed balance). For $\xi \in \{0, 1\}^{\Lambda}$ put $\tilde{\xi} = \{x \in \Lambda : \xi_x = 1\}$ and we denote by $\mathbf{n}(\xi) = |\tilde{\xi}|$ the number of elements of $\tilde{\xi}$ (that is, $|A|$ denotes the cardinality of A). Observe that for each $0 < N < |\Lambda|$ there exists a unique $\lambda = \lambda(N, \alpha) \in \mathbb{R}$ such that $|\mu| = |\mu|^{\alpha,\lambda}$ verifies $|\mu|(\mathbf{n}) = N$ (see Ref. 4).

1.2. Gibbs Measures for the Multicolour Lattice Gas

Let $\xi \in \{0,1\}^\Lambda$. In the sequel a configuration $\bar{\eta} \in I^{\tilde{\xi}}$ is naturally identified with $\eta \in I_0^\Lambda$ where $\eta_x = \bar{\eta}_x \mathbf{1}(x \in \tilde{\xi})$.

Let $\mathcal{P}(I)$ be the simplex of probability measures on I and $p = (p_i : i \in I) \in \mathcal{P}(I)$ be one of these measures. Let $K \subseteq \Lambda$, μ_p^K denotes the product measure on I^K given by

$$\mu_p^K(\eta) = \prod_{x \in K} p_{\eta_x} \ , \quad \text{for} \quad \eta = (\eta_x : x \in K) \in I^K \ .$$

If $\xi \in \{0,1\}^\Lambda$ we set $\mu_p^\xi := \mu_p^{\tilde{\xi}}$, so that indeed μ_p^ξ is the product measure on $I^{\tilde{\xi}}$ with marginals $\mu_p^\xi(\eta : \eta_x = i) = p_i$ for $x \in \tilde{\xi}$. Then

$$\mu_p^\xi(\eta) = \prod_{i \in I} p_i^{\mathbf{n}_i(\eta)} \ , \quad \text{for} \quad \eta \in I^{\tilde{\xi}} \ ,$$

where $\mathbf{n}_i(\eta) = |\{x \in \tilde{\xi} : \eta_x = i\}|$.

Now we are able to define our set of Gibbs measures. The probability measure μ on I_0^Λ is a Gibbs one if it satisfies

$$\mu(\eta \in I_0^\Lambda : |\eta| = \xi) = |\mu|(\xi) \text{ for every } \xi \in \{0,1\}^\Lambda \qquad (H1)$$

and

$$\exists p \in \mathcal{P}(I) \text{ such that } \mu^\xi := \mu(\cdot \mid |\eta| = \xi) = \mu_p^{\tilde{\xi}} \text{ for every } \xi \in \{0,1\}^\Lambda. \qquad (H2)$$

We will keep the term Gibbs measure only for those measures verifying $(H1)$ and $(H2)$. We notice that they are symmetric and invariant with respect to the dynamics defined in Refs. 1 and 2, which is an extension of the monocolour one, where the rate of change $c_{x,x+e}^\alpha(\eta) = f_e(\alpha_x, |\eta_x|, \alpha_{x+e}, |\eta_{x+e}|)$. But they are not the unique measures satisfying these properties. Indeed, if μ verifies $(H1)$ and for some probability measure Q defined on the simplex $\mathcal{P}(I)$ it holds

$$\mu(\eta) = \int_{\mathcal{P}(I)} \mu_p^\xi(\eta) dQ(p) = \int_{\mathcal{P}(I)} \prod_{i \in I} p_i^{\mathbf{n}_i(\eta)} dQ(p) \quad \text{for} \quad \eta \in I^{\tilde{\xi}} \ ,$$

then μ is symmetric and so invariant for the dynamics defined in Refs. 1 and 2. We recall that such μ is exchangeable, that is invariant under coordinate permutations.

When μ is a Gibbs measure μ^ξ is a measure on I_0^Λ that gives strictly positive weights to the configurations vanishing at $\Lambda \setminus \tilde{\xi}$ and taking values in I at $\tilde{\xi}$. So, with the above identification μ^ξ can be also seen as a probability measure on $I^{\tilde{\xi}}$. With this notation $\mu^{|\eta|}(\cdot) = \mu(\cdot \mid |\eta|)$ is a measure concentrated on the set

$\{\eta' : |\eta'| = |\eta|\}$. It follows that for every $g : I_0^\Lambda \to \mathbb{R}$

$$\mu(g) = \sum_{\eta \in I_0^\Lambda} g(\eta)\mu(\eta) = \sum_{\eta \in I_0^\Lambda} g(\eta)\mu^{|\eta|}(\eta)|\mu|(|\eta|) = \sum_{\xi \in \{0,1\}^\Lambda} \left(\sum_{\eta : |\eta| = \xi} g(\eta)\mu^\xi(\eta) \right) |\mu|(\xi).$$

Therefore

$$\mu(g) = |\mu|(S^\mu(g)) \quad \text{with} \quad S^\mu(g)(\xi) = \sum_{\eta : |\eta| = \xi} g(\eta)\mu^\xi(\eta) \quad \text{for} \quad \xi \in \{0, 1\}^\Lambda. \tag{1.3}$$

Notice that $S^\mu(g)$ depends on μ via the set of conditional measures (μ^ξ: $\xi \in \{0, 1\}^\Lambda$).

1.3. Canonical Gibbs Measures for the Multicolour Lattice Gas

Let $0 < N < |\Lambda|$ be fixed. Recall $\mathbf{n}(\xi) = |\tilde{\xi}|$. Let $(\mathbf{n} = N) = \{\xi \in \{0, 1\}^\Lambda : \mathbf{n}(\xi) = N\}$ and μ is a Gibbs measure for the multicolour lattice gas. The monocolour canonical Gibbs measure is the following conditional measure on $\{0, 1\}^\Lambda$:

$$|\nu|(\cdot) = |\mu|(\cdot \mid \mathbf{n} = N)$$

Let $(N_i > 0 : i \in I)$ be fixed with $\sum_{i \in I} N_i = N$. Recall $\mathbf{n}_i(\eta) = |\{x \in \tilde{\xi} : \eta_x = i\}|$. Let $(\mathbf{n}_i = N_i : i \in I) = \{\eta \in I_0^\Lambda : \mathbf{n}_i(\eta) = N_i, i \in I\}$. The canonical Gibbs measure for the multicolour lattice gas is the following conditional measure on I_0^Λ:

$$\nu(\cdot) = \mu(\cdot \mid \mathbf{n}_i = N_i : i \in I).$$

Before giving our main result we need the following background. For $\eta \in I_0^\Lambda$ and $\Delta \subseteq \Lambda$ put $\eta_\Delta = (\eta_x : x \in \Delta)$. Let $g : I_0^\Lambda \to \mathbb{R}$. The function g depends on Δ if for any couple $\eta, \eta' \in I_0^\Lambda$ such that $\eta_\Delta = \eta'_\Delta$, it holds $g(\eta) = g(\eta')$. This is equivalent to the existence of a function $g_\Delta : I_0^\Delta \to \mathbb{R}$ such that $g(\eta) = g_\Delta(\eta_\Delta)$. If g depends on Δ_1 and on Δ_2, then it is easy to show that g depends on $\Delta_1 \cap \Delta_2$. Let $\mathbf{D}(g) = \{\Delta \subseteq \Lambda : g \text{ depends on } \Delta\}$, and put $\Delta(g) = \cap_{\Delta \in \mathbf{D}(g)} \Delta$. Then g depends on $\Delta(g)$ and this set is the smallest one verifying this property and we call it the support of g. So, $\eta_{\Delta(g)} = \eta'_{\Delta(g)}$ implies $g(\eta) = g(\eta')$ and by abuse of notation we put $g(\eta) = g(\eta_{\Delta(g)})$.

Lemma 1.1. *Let μ be a Gibbs measure. Then for every $g : I_0^\Lambda \to \mathbb{R}$ it holds $\Delta(S^\mu(g)) \subseteq \Delta(g)$ and $||S^\mu(g)||_\infty \leq ||g||_\infty$.*

Proof: From $S^\mu(g)(\xi) = \sum_{\eta : |\eta| = \xi} g(\eta)\mu^\xi(\eta)$ the inequality between the norms $|| \cdot ||_\infty$ is direct. Let us show the inclusion of the supports. Observe that $\mu^\xi = \mu_p^\xi$ for all $\xi \in \{0, 1\}^\Lambda$, for a fixed probability vector $p = (p_i : i \in I) \in \mathcal{P}(I)$. Put

$\Delta = \Delta(g)$ and partition $\eta = (\eta_\Delta, \eta_{\Lambda\setminus\Delta})$, $\xi = (\xi_\Delta, \xi_{\Lambda\setminus\Delta})$. Since μ_p^ξ is a product measure and $g(\eta) = g(\eta_\Delta)$ we have

$$\begin{aligned} S^\mu(g)(\xi) &= \sum_{\eta_\Delta : |\eta_\Delta| = \xi_\Delta} g(\eta_\Delta)\mu_p^{\xi_\Delta}(\eta_\Delta) \left(\sum_{\eta_{\Lambda\setminus\Delta} : |\eta_{\Lambda\setminus\Delta}| = \xi_{\Lambda\setminus\Delta}} \mu_p^{\xi_{\Lambda\setminus\Delta}}(\eta_{\Lambda\setminus\Delta}) \right) \\ &= \sum_{\eta_\Delta : |\eta_\Delta| = \xi_\Delta} g(\eta_\Delta)\mu_p^{\xi_\Delta}(\eta_\Delta) \,. \end{aligned}$$

Therefore $S^\mu(g)(\xi)$ only depends on ξ_Δ, and the result is shown. □

Now we can announce our main result.

Theorem 1.2. *Let $p = (p_i : i \in I)$ with $p_i = N_i/N, i \in I$, μ be a Gibbs measure such that $\mu^\xi = \mu_p^\xi$, and $|\mu|(\mathbf{n}) = \mathrm{N}$. Suppose that* $\min\{N_i : i \in I\} := \psi(N) \geq 1$ *for every N. Then, for any $L > 1$ fixed, there exists constant $C = C(|I|, L)$ such that for all $g : I_0^\Lambda \to \mathbb{R}$ verifying $|\Delta(g)| \leq \psi(N)/L$ it holds*

$$||\nu|(S^\mu(g)) - \nu(g)\,| \leq C||g||_\infty \frac{|\Delta(g)|}{\psi(N)} \,. \tag{1.6}$$

More precisely

$$||\nu|(S^\mu(g)) - \nu(g)\,| \leq \frac{C||g||_\infty}{\psi(N)} \left(\sum_{\xi \in \{0,1\}^\Lambda : \mathbf{n}(\xi) = N} \big(|\Delta(g) \cap \tilde{\xi}|\big)|\mu|(\xi \mid \mathbf{n} = N) \right) \tag{1.7}$$

In particular, we get that, in the framework of the Theorem, there exists a constant $D = D(|I|, L)$ such that

$$||\nu|(S^\mu(g)) - \nu(g)| \leq D\frac{||g||_\infty |\Delta(g)|}{\psi(N)} \tag{1.8}$$

From (1.3), $\mu(g) = |\mu|(S^\mu(g))$. We deduce

$$\mu(g) - \nu(g) = |\mu|(S^\mu(g)) - |\nu|(S^\mu(g)) + |\nu|(S^\mu(g)) - \nu(g). \tag{1.9}$$

Now, Lemma 1.1. gives the inclusion $\Delta(S^\mu(g)) \subseteq \Delta(g)$. Then, from Lemma A.4 in Ref. 4, we get that: for any $\gamma \in (0, 1)$ and any g with support $\Delta(g) \subset \Lambda$ and such that $|\Delta(g)| \leq |\Lambda|^{1-\gamma}$, for large Λ it holds

$$|\mu|(S^\mu(g)) - |\nu|(S^\mu(g)) \leq C\,||g||_\infty \frac{|\Delta(g)|}{|\Lambda|}. \tag{1.10}$$

Therefore, by combining Theorem 1.2 with (1.8), (1.9) and (1.10), we get the following result.

Corollary 1.3. *Let* $p = (p_i : i \in I)$ *with* $p_i = N_i/N$, $i \in I$, *and* μ *a Gibbs measure such that* $\mu^\xi = \mu_p^\xi$. *Suppose that there exists* $\delta > 0$ *such that* $\min\{N_i : i \in I\} \geq \delta N$ *for every* N. *Then, for any* $\gamma \in (0, 1)$ *there exists constant* $C' = C'(|I|, \gamma, \delta)$ *such that for all* $g : I_0^\Lambda \to \mathbb{R}$ *verifying* $|\Delta(g)| \leq N^{1-\gamma}$ *it holds*

$$|\mu(g) - \nu(g)| \leq C' \frac{||g||_\infty |\Delta(g)|}{N} \ . \tag{1.12}$$

The rest of this paper is consecrated to the proof of Theorem 1.2. We point out that the main technical point is the control of $|\nu|(S^\mu g) - \nu(g)$. From (2.1) and (2.3) below, we are led to study a quotient that is dominated by the usual entropy estimates, see the proof of Lemma 2.6.

2. PROOF OF THE MAIN RESULT

Let μ be a Gibbs measure for the multicolour lattice gas. Recall that the monocolour canonical Gibbs measure is defined on $\{0, 1\}^\Lambda$ by

$$|\nu|(\cdot) = |\mu|(\cdot \mid \mathbf{n} = N) \ ,$$

so $|\nu|(\xi) = \mathbf{1}(\mathbf{n}(\xi) = N) \frac{|\mu|(\xi)}{|\mu|(\mathbf{n}=N)}$ for $\xi \in \{0, 1\}^\Lambda$. Therefore,

$$|\mu|(f) - |\nu|(f) = \sum_{\xi \in \{0,1\}^\Lambda} f(\xi)|\mu|(\xi) \left(1 - \frac{\mathbf{1}(\mathbf{n}(\xi) = N)}{|\mu|(\mathbf{n} = N)}\right) \text{ for } f : \{0, 1\}^\Lambda \to \mathbb{R} \ .$$

The canonical Gibbs measure for the multicolour lattice gas is defined on I_0^Λ by

$$\nu(\cdot) = \mu(\cdot \mid \mathbf{n}_i = N_i : i \in I).$$

Hence,

$$\nu(g) = \sum_{\eta \in I_0^\Lambda} g(\eta)\mu(\eta) \frac{\mathbf{1}(\mathbf{n}_i(\eta) = N_i : i \in I)}{\mu(\mathbf{n}_i = N_i : i \in I)} \quad \text{for} \quad g : I_0^\Lambda \to \mathbb{R} \ ,$$

so,

$$\mu(g) - \nu(g) = \sum_{\eta \in I_0^\Lambda} g(\eta)\mu(\eta) \left(1 - \frac{\mathbf{1}(\mathbf{n}_i(\eta) = N_i : i \in I)}{\mu(\mathbf{n}_i = N_i : i \in I)}\right) \ .$$

Let us first establish some general relations which do not depend on the fact that p verifies the requirements of the Theorem. The hypothesis on p will be only used at the end of the proof, in Lemma 2.3.

We have

$$|\nu|(S^\mu(g)) = \sum_{\xi \in \{0,1\}^\Lambda} \left(\sum_{\eta : |\eta| = \xi} g(\eta)\mu^\xi(\eta) \right) \frac{\mathbf{1}(\mathbf{n}(\xi) = N)}{|\mu|(\mathbf{n} = N)} |\mu|(\xi) \ .$$

Since $\{\eta \in I_0^\Lambda : \mathbf{n}_i = N_i, i \in I\} \subseteq \{\eta \in I_0^\Lambda : \mathbf{n}(|\eta|) = N\}$ and $\mu(\eta : \mathbf{n}(|\eta|) = N) = |\mu|(\xi : \mathbf{n}(\xi) = N)$, we deduce

$$\nu(g) = \sum_{\xi \in \{0,1\}^\Lambda} \left(\sum_{\eta : |\eta| = \xi} g(\eta)\mu^\xi(\eta) \frac{\mathbf{1}(\mathbf{n}_i(\eta) = N_i : i \in I)}{\mu(\mathbf{n}_i = N_i, i \in I \mid \mathbf{n} = N)} \right) \frac{\mathbf{1}(\mathbf{n}(\xi) = N)}{|\mu|(\mathbf{n} = N)} |\mu|(\xi).$$

Therefore

$$|\nu|(S^\mu(g)) - \nu(g) = \sum_{\xi \in \{0,1\}^\Lambda : \mathbf{n}(\xi) = N} \left(\Psi_1^g(\xi) - \Psi_2^g(\xi) \right) |\mu|(\xi \mid \mathbf{n} = N) \,, \qquad (2.1)$$

where

$$\Psi_1^g(\xi) = \sum_{\eta : |\eta| = \xi} g(\eta)\mu^\xi(\eta) \quad \text{and} \quad \Psi_2^g(\xi)$$

$$= \sum_{\eta : |\eta| = \xi, \mathbf{n}_i(\eta) = N_i, i \in I} g(\eta) \frac{\mu^\xi(\eta)}{\mu(\mathbf{n}_i = N_i, i \in I \mid \mathbf{n} = N)} \,.$$

From now on, we fix $\mu^\xi = \mu_p^\xi$ with $p = (p_i : i \in I)$ a probability vector in I. Since for every $\eta \in I_0^\Lambda$ such that $\mathbf{n}_i(\eta) = N_i$ for $i \in I$, it holds $\mu^\xi(\eta) = \prod_{i \in I} p_i^{N_i}$, we get

$$\Psi_2^g(\xi) = \frac{\prod_{i \in I} p_i^{N_i}}{\mu(\mathbf{n}_i = N_i, i \in I \mid \mathbf{n} = N)} \sum_{\eta : |\eta| = \xi, \mathbf{n}_i(\eta) = N_i, i \in I} g(\eta) \,.$$

Let $\binom{N}{N_i, i \in I} = \frac{N!}{\prod_{i \in I} N_i!}$. We have

$$\mu(\mathbf{n}_i = N_i, i \in I \mid \mathbf{n} = N) = \binom{N}{N_i, i \in I} \prod_{i \in I} p_i^{N_i} \,,$$

so,

$$\frac{\prod_{i \in I} p_i^{N_i}}{\mu(\mathbf{n}_i = N_i, i \in I \mid \mathbf{n} = N)} = \binom{N}{N_i,\ i \in I}^{-1} .$$

Then,

$$\Psi_2^g(\xi) = \sum_{\eta : |\eta| = \xi, \mathbf{n}_i(\eta) = N_i, i \in I} g(\eta)\rho(\eta) \quad \text{with} \quad \rho(\eta) = \binom{N}{N_i,\ i \in I}^{-1} .$$

In the sequel, for every $\xi \in \{0,1\}^\Lambda$ with $\mathbf{n}(\xi) = N$ we put $\Delta_\xi = \Delta(g) \cap \tilde{\xi}$ and $\partial_\xi = \tilde{\xi} \setminus \Delta(g)$. Then $\tilde{\xi} = \Delta_\xi \cup \partial_\xi$ is a partition. Every $\eta \in I_0^\Lambda$ with $|\eta| = \xi$ can be represented by $\eta = (\eta_{\Delta_\xi}, \eta_{\partial_\xi})$. In fact in the complement we have $\eta_x = 0$ for $x \in \Lambda \setminus \tilde{\xi}$. With this notation and since $g(\eta) = g(\eta_{\Delta(g)})$, we obtain

$$g(\eta) = g(\eta_{\Delta_\xi}) \text{ for every } \eta \in I_0^\Lambda \text{ with } |\eta| = \xi \ .$$

Then, by using μ_p^ξ is a product measure we get

$$\Psi_1^g(\xi) = \sum_{\eta:|\eta|=\xi} g(\eta)\mu_p^\xi(\eta) = \sum_{\eta_{\Delta_\xi} \in I^{\Delta_\xi}} \sum_{\eta_{\partial_\xi} \in I^{\partial_\xi}} g(\eta_{\Delta_\xi})\mu_p^{\xi_{\Delta(g)}}(\eta_{\Delta_\xi})\mu_p^{\xi_{\Lambda\setminus\Delta(g)}}(\eta_{\partial_\xi})$$
$$= \sum_{\eta_{\Delta_\xi} \in I^{\Delta_\xi}} g(\eta_{\Delta_\xi})\mu_p^{\xi_{\Delta(g)}}(\eta_{\Delta_\xi}) \ .$$

In a similar way, we find

$$\Psi_2^g(\xi) = \sum_{\eta_{\Delta_\xi} \in I^{\Delta_\xi}} g(\eta_{\Delta_\xi}) \sum_{\eta_{\partial_\xi} \in I^{\partial_\xi}:\mathbf{n}_i(\eta_{\partial_\xi})=N_i-\mathbf{n}_i(\eta_{\Delta_\xi}),\ i\in I} \binom{N}{N_i,\ i \in I}^{-1}$$
$$= \sum_{\eta_{\Delta_\xi} \in I^{\Delta_\xi}} g(\eta_{\Delta_\xi}) \binom{N - |\Delta_\xi|}{N_i - \mathbf{n}_i(\eta_{\Delta_\xi}),\ i \in I} \binom{N}{N_i,\ i \in I}^{-1} .$$

Then,

$$\Psi_1^g(\xi) - \Psi_2^g(\xi) = \sum_{\eta_{\Delta_\xi} \in I^{\Delta_\xi}} g(\eta_{\Delta_\xi}) \left(\mu_p^{\xi_{\Delta(g)}}(\eta_{\Delta_\xi}) - \binom{N - |\Delta_\xi|}{N_i - \mathbf{n}_i(\eta_{\Delta_\xi}),\ i \in I} \times \binom{N}{N_i,\ i \in I}^{-1} \right) \tag{2.2}$$

and we obtain,

$$|\Psi_1^g(\xi) - \Psi_2^g(\xi)| \le ||g||_\infty M_\xi \text{ with} \tag{2.3}$$
$$M_\xi = \sum_{\eta_{\Delta_\xi} \in I^{\Delta_\xi}} \left| \mu_p^{\xi_{\Delta(g)}}(\eta_{\Delta_\xi}) - \binom{N - |\Delta_\xi|}{N_i - \mathbf{n}_i(\eta_{\Delta_\xi}),\ i \in I} \binom{N}{N_i,\ i \in I}^{-1} \right| .$$

We observe that $M_\xi = ||q - r||$ is the total variation of two probability measures q and r defined on I^{Δ_ξ} by

$$q(\eta_{\Delta_\xi}) = \mu_p^{\xi_{\Delta(g)}}(\eta_{\Delta_\xi}) \text{ and } r(\eta_{\Delta_\xi}) = \binom{N - |\Delta_\xi|}{N_i - \mathbf{n}_i(\eta_{\Delta_\xi}),\ i \in I} \binom{N}{N_i,\ i \in I}^{-1} .$$

Therefore,

$$M_\xi = 2\left(\sum_{\eta_{\Delta_\xi}\in I^{\Delta_\xi}\,:\,r(\eta_{\Delta_\xi})>q(\eta_{\Delta_\xi})} (r(\eta_{\Delta_\xi}) - q(\eta_{\Delta_\xi}))\right) .$$

Let us denote $\ell_\xi = |\Delta_\xi|$ and $\mathcal{J}_\xi = \{\vec{l} = (l_i : i \in I) \in \mathbb{N}^I : l_i \geq 0 \text{ for } i \in I, \sum_{i\in I} l_i = \ell_\xi\}$. To every η_{Δ_ξ} we associate $\vec{l}(\eta_{\Delta_\xi}) = (l_i : i \in I) \in \mathcal{J}_\xi$ with $l_i = \mathbf{n_i}(\eta_{\Delta_\xi}) = |\{\mathbf{x} \in \Delta_\xi : \eta_\mathbf{x} = \mathbf{i}\}|$. We observe that $q(\eta_{\Delta_\xi}) = q(\vec{l}(\eta_{\Delta_\xi}))$ and $r(\eta_{\Delta_\xi}) = r(\vec{l}(\eta_{\Delta_\xi}))$. Then, by defining

$$q(\vec{l}) = \prod_{i\in I} p_i^{l_i} \text{ and } r(\vec{l}) = \binom{N-\ell_\xi}{N_i - l_i,\, i \in I}\binom{N}{N_i,\, i\in I}^{-1} \tag{2.4}$$

we find,

$$M_\xi = 2\left(\sum_{\vec{l}\in\mathcal{J}_\xi:\, r(\vec{l})>q(\vec{l})} q(\vec{l})\left(\frac{r(\vec{l})}{q(\vec{l})} - 1\right)\right) . \tag{2.5}$$

The proof of the Theorem 1.2 will be deduced from the following result.

Lemma 2.3. *Let $p = (p_i : i \in I)$ with $p_i = N_i/N, i \in I$. Assume that*

$$\min\{N_i : i \in I\} := \psi(N) \geq 1, \text{ and } |\Delta(g)| \leq L^{-1}\psi(N)$$

for some $L > 1$. Then

$$M_\xi \leq e^{\Theta^*(\xi)}\left(1 - \frac{\ell_\xi}{\psi(N)}\right)^{-|I|/2} - 1 \tag{2.7}$$

where

$$\Theta^*(\ell_\xi) = \frac{\ell_\xi}{11N(N-\ell_\xi)} \tag{2.8}$$

Proof: From relation (2.5) it suffices to show that

$$\max\left\{\frac{r(\vec{l})}{q(\vec{l})} : \vec{l} \in \mathcal{J}_\xi\right\} \leq e^{\Theta^*(\xi)}\left(1 - \frac{\ell_\xi}{\psi(N)}\right)^{-|I|/2} . \tag{2.9}$$

From (2.4) we have

$$\frac{r(\vec{l})}{q(\vec{l})} = \frac{N^{\ell_\xi}(N-\ell_\xi)!\prod_{i\in I} N_i!}{N!\prod_{i\in I}\left((N_i - l_i)!N_i^{l_i}\right)} . \tag{2.10}$$

By using the Stirling formula

$$m! = e^{m(\log m - 1)}(2\pi m)^{1/2} e^{\theta(m)} \text{ with } \frac{1}{12m+1} < \theta(m) < \frac{1}{12m} ,$$

we find,

$$\frac{r(\vec{l})}{q(\vec{l})} = Z(\vec{l}) e^{H(\vec{l})} e^{\Theta(\vec{l})} ,$$

where

$$Z(\vec{l}) = \left(\frac{(N-\ell_\xi)\prod_{i\in I} N_i}{N\prod_{i\in I}(N_i - l_i)} \right)^{1/2} = \left(1 - \frac{\ell_\xi}{N}\right)^{1/2} \prod_{i\in I} \left(1 - \frac{l_i}{N_i}\right)^{-1/2} ,$$

$$\Theta(\vec{l}) = \theta(N-\ell_\xi) - \theta(N) + \sum_{i\in I} (\theta(N_i) - \theta(N_i - l_i)) ,$$

$$H(\vec{l}) = \ell_\xi \log N + (N-\ell_\xi)(\log(N-\ell_\xi) - 1) - N(\log N - 1)$$
$$+ \sum_{i\in I} (N_i(\log N_i - 1) - (N_i - l_i)(\log(N_i - l_i) - 1) - l_i \log N_i).$$

Then, $H(\vec{l}) = (N-\ell_\xi)\log(1 - \frac{\ell_\xi}{N}) - \sum_{i\in I}(N_i - l_i)\log(1 - \frac{l_i}{N_i})$. Let us consider the maximization problem

$$\max \left(H(\vec{l}) : \vec{l} \in \ell_\xi \mathcal{P}(I) \right) \text{ where } \ell_\xi \mathcal{P}(I) = \{\vec{l} \in \mathbb{R}_+^I : l_i \geq 0, i \in I, \sum_{i\in I} l_i = \ell_\xi\}.$$

It is easy to show that the maximum is attained at $l_i/N_i = \chi$ a constant, then by the constraints $\sum_{i\in I} l_i = \ell_\xi$ we find that $l_i/N_i = \ell_\xi/N$ for all $i \in I$, and at this point $H(\vec{l}) = 0$. Since $\mathcal{J}_\xi \subset \ell_\xi \mathcal{P}(I)$, we deduce $\max \left(H(\vec{l}) : \vec{l} \in \mathcal{J}_\xi \right) \leq 0$.

On the one hand from $\min(N_i : i \in I) \geq \psi(N)$ and $l_i \leq \ell_\xi$, we have

$$\prod_{i\in I} \left(1 - \frac{l_i}{N_i}\right)^{-1/2} \leq (1 - \frac{\ell_\xi}{\psi(N)})^{-|I|/2}$$

then $Z(\vec{l}) \leq (1 - \frac{\ell_\xi}{\psi(N)})^{-|I|/2}$. A simple computation shows that

$$\theta(N-\ell_\xi) - \theta(N) \leq \frac{1}{12(N-\ell_\xi)} - \frac{1}{12N+1} = \frac{1+12\ell_\xi}{12(N-\ell_\xi)(12N+1)}$$
$$\leq \frac{13\ell_\xi}{144(N-\ell_\xi)N} \leq \frac{\ell_\xi}{11(N-\ell_\xi)N}$$

On the other hand for all i, $\theta(N_i) - \theta(N_i - l_i) = 0$ for $l_i = 0$, and for $l_i \geq 1$,

$$\theta(N_i) - \theta(N_i - l_i) \leq \frac{-12l_i + 1}{12N_i(12(N_i - l_i) + 1)}) \leq 0 .$$

From the latter estimates we get

$$\Theta(\vec{l}) \leq \frac{\ell_\xi}{11N(N-\ell_\xi)} = \Theta^*(\ell_\xi) \,.$$

This shows (2.9), and the Lemma follows. □

Let us finish the proof of Theorem 1.2. From relations (2.1), (2.3) and (2.7), and by using the hypothesis of the Theorem, we obtain

$$||\nu|(S^\mu(g)) - \nu(g)| \leq \sum_{\xi\in\{0,1\}^\Lambda:\, \mathbf{n}(\xi)=N} ||g||_\infty \, M_\xi \, |\mu|(\xi \mid \mathbf{n}=N)$$

$$\leq ||g||_\infty \sum_{\xi\in\{0,1\}^\Lambda:\, \mathbf{n}(\xi)=\mathbf{N}} \left(e^{\Theta^*(\xi)} \left(1 - \frac{\ell_\xi}{\psi(N)}\right)\right)^{-|I|/2} - 1 \right) |\mu|(\xi \mid \mathbf{n}=\mathrm{N}) \,.$$

From $|\Delta(g)| \leq L^{-1}\psi(N)$ and $\psi(N) \leq N$ we have $\ell_\xi \leq L^{-1}N$. Below we use $(1-x)^{-1} \leq 1 + hx$ when $0 \leq x \leq (h-1)/h$, to deduce that

$$\Theta^*(\ell_\xi) \leq \frac{(2L-1)\ell_\xi}{11(L-1)N^2}.$$

From the inequality $\ell_\xi \leq L^{-1}\psi(N)$, we get

$$\left(1 - \frac{\ell_\xi}{\psi(N)}\right)^{-|I|/2} = \left((1 - \frac{\ell_\xi}{\psi(N)})^{-1}\right)^{|I|/2} \leq \left(1 + \frac{L\ell_\xi}{(L-1)\psi(N)}\right)^{\frac{|I|}{2}}$$

$$\leq 1 + \frac{|I|2^{\frac{|I|}{2}-1}L\ell_\xi}{2(L-1)\psi(N)}$$

Finally

$$\left(e^{\Theta^*(\xi)} \left(1 - \frac{\ell_\xi}{\psi(N)}\right)^{-|I|/2} - 1 \right) = \frac{|I|2^{\frac{|I|}{2}-1}L\ell_\xi}{2(L-1)\psi(N)} + O\left(\frac{(2L-1)\ell_\xi}{11(L-1)N^2}\right)$$

The theorem follows directly from all these bounds.

ACKNOWLEDGMENTS

S.M. acknowledges support from the Nucleus Millennium Information and Randomness P04-069-F and he is indebted for the hospitality at the Laboratoire Paul Painlevé, Université des Sciences et Technologies de Lille, where this work started. The authors also thank the referees for their comments which improve our first version.

REFERENCES

1. A. Dermoune, and P. Heinrich, A small step towards the hydrodynamic limit of a colored disordered lattice gas, *C. R. Acad. Sci. Paris, Ser. I* **339**:507–511 (2004).
2. A. Dermoune, and P. Heinrich, Equivalence of ensembles for colored particles in a disordered lattice gas, *Markov Processes Relat. Fields* **11**:405–424 (2005).
3. A. Faggionato, *Hydrodynamic limit of a disordered system*, Ph.D. Thesis, Scuola Normale Superiore di Pisa, *http://mpej.unige.ch/mp_arc/c/03/03-37.ps.gz*, (2002).
4. A. Faggionato, and F. Martinelli, Hydrodynamic limit of a disordered lattice gas, *Probab. Theory Relat. Fields* **127**:535–608 (2003).
5. F. Rezakhanlou, Propagation of chaos for symmetric simple exclusions, *Comm. on Pure and Appl. Math.* **XLVII**:943–957 (1994).